Composite Electrodynamics

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This work is dedicated to I.M. Gelfand
upon the occasion of his 75th birthday
with friendship and admiration

Abstract. *This paper solidifies the foundations for a singleton theory of light, first proposed two years ago. This theory is based on a pure gauge coupling of the scalar singleton field to the electromagnetic current. Like quarks, singletons are essentially unobservable. The field operators are not local observables and therefore need not commute for spacelike separation. This opens up possibilities for generalized statistics, just as is the case for quarks. It then turns out that a pure gauge coupling, in which* $\partial_\mu\phi(x)$ *couples to the conserved current* $j^\mu(x)$ *, generates real interactions – the effective theory is precisely ordinary electrodynamics in de Sitter space. Here we improve our theory and explain it in much more detail than before, adding two new results. (1) The concept of normal ordering in a theory with unconventional statistics is worked out in detail. (2) We have discovered the natural way of including both photon helicities. Quantization, it may be noted, is a study in representation theory of certain infinite-dimensional, nilpotent Lie algebras, of which the Heisenberg algebra is the prototype.*

INTRODUCTION

Some of the curious features of singleton kinematics were pointed out long ago [1, 2]. Briefly, the extreme paucity of states is a strong indication of the impossibility of direct observation of one-singleton states. Not less remarkable is the fact that all two-singleton states are massless. On this purely kinematical basis we pointed out, in 1978, that all

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massless particles can be interpreted as being composed of two singletons. (This idea, a revival of the old neutrino theory of light, has recently been found to fit in very nicely with the development of super membranes [3]).

Soon afterwards, we took the first steps towards the development of a singleton quantum field theory, with the discovery [4] of a *full fledged gauge theory* of a new kind, in which no vector fields but only scalar and spinor fields are involved. The suggestion of non-observability alluded to above is henceforth strongly reinforced by the requirement of unitarity. Indeed, *this field theory is unitary only if all interactions are gauge invariant*; and by a gauge transformation *the fields can be transformed to zero* in every compact region of de Sitter space. (Later, when singletons showed up in higher dimensional supergravities [5] they did, in fact, turn out to satisfy this condition).

This situation seemed at first to lead a dead end, for two reasons. First, the newly discovered gauge invariance seemed to prohibit local interactions (only interactions at infinity were allowed). Secondly, the composite objects did not obey Bose-Einstein statistics, even approximately. This is a problem, though often overlooked, in all composite theories, but it is especially acute when, as in our case, the binding energy vanishes. We had assumed, of course, that the singletons themselves obey conventional quantum statistics. This impasse lasted until 1986.

The key to a solution of both problems is to realize that objects that cannot be observed locally (singletons, preons, and quarks) have no duty to obey the usual statistics, since their fields need not commute at spacelike distances. We noticed that the alleged triviality of an interaction of the form

$$\int j^\mu(x)\,\partial_\mu\phi(x)\,\mathrm{d}x, \tag{1.1}$$

in which j is a conserved current and ϕ is a scalar field, can be proved only if the normal ordered commutator of $\phi(x)$ with $\partial_\mu\phi(x)$ is assumed to vanish – as is the case with the Bose-Einstein quantization rules. Indeed, the effective interaction generated by (1.1) is

$$\int j^\mu(x) : [\phi(x), \partial_\mu\phi(x)] : .$$

So what was needed was a quantization scheme in which this vector field is different from zero. We shall enter into the details later, for one purpose of this paper is to make up for the inadequacies of our brief Letter [6]. The main point, however, is that QED was reformulated as singleton field theory, photons being identified with (antisymmetric) states of two singletons, and the full dynamics of photons and electrons being generated by the interaction (1.1) between electrons and singletons.

The new insight that is offered here concerns mainly three points: clarification and completion of the quantum rules (the properties of normal ordering, especially), the inclusion into the theory of two helicities, as required by conformal invariance (in de Sitter space there is a version of QED that contains only one photon helicity; this version is not conformally invariant), and the possibility of CP violation.

The structure of the S-matrix is understood modulo the possibility of anomalies. Anomalies are by no means to be considered as a nuisance, for the theory embodies a latent CP violation that needs only an anomaly to make it real. (Last section).

Elsewhere [7], we have discussed quantization, including the new type of quantum field theory that is the subject of the present paper, from the point of view of the representation theory of infinite dimensional, nilpotent Lie algebras. Those that appear in physics have a complex structure that leads naturally to concept of highest weight representations, or generalized Fock space representations, that can be fully classified.

We believe that the proper foundations have now been laid down, on which to build an extension of the singleton-composite picture to non-Abelian gauge theories.

2. SINGLETON GAUGE THEORY

The free field of (Bose) singleton gauge theory is a scalar field that satisfies the dipole equation [8]

$$\left(\Box - \frac{5}{4}\rho\right)^2 \phi(x) = 0. \tag{2.1}$$

The second order differential operator $\Box$ is the d'Alembertian constructed from the de Sitter metric, and the equation is invariant under the natural action of $\mathrm{SO}(3,2)$ on the de Sitter hyperboloid (more precisely: its double covering). In particular, the Lie algebra $\mathrm{so}(3,2)$ acts on the space of solutions with positive energy and finite angular momentum; this action is an indecomposable representation with the following Gupta-Bleuler triplet structure

$$D(5/2,0) \to D(1/2,0) \to D(5/2,0). \tag{2.2}$$

Since each component is irreducible, there are precisely two proper, invariant subspaces. The larger subspace is defined by the Lorentz condition

$$\left(\Box - \frac{5}{4}\rho\right) \phi(x) = 0 \tag{2.3}$$

and carries the subrepresentation $D(1/2,0) \to D(5/2,0)$. The smaller invariant subspace carries only $D(5/2,0)$ and consists of gauge modes. The quotient $D(1/2,0)$ is the space of physical states.

The Lagrangian associated with Eq. (2.1) must be constructed with careful attention to surface terms arising from integration by parts, since the solutions fall off very slowly at infinity. Is is (with $u = -5\rho/4$)

$$\mathcal{L}[\phi, b] = \int d^4x\,(-g)^{\frac{1}{2}} \left(g^{\mu\nu}\phi_\mu b_\nu - u\phi b + \frac{1}{2}bb\right) + \tag{2.4}$$
$$+ \frac{1}{3}\int d^4x\,\Box(-g)^{\frac{1}{2}} \left(g^{\mu\nu}\phi_\mu\phi_\nu + \frac{1}{4}\rho\phi\phi - \frac{1}{2}\phi b\right).$$

The field b plays the role of the Nakanishi-Lautrup field of this gauge theory. A BRST extension is effected by adding [9]

$$\mathcal{L}[c, d] = \int d^4x\,(-g)^{\frac{1}{2}}(-g^{\mu\nu}c_\mu d_\nu + ucd) \tag{2.5}$$

where c and d are the Fadde'ev-Popov ghost and antighost fields. The BRST transformation, under which the total Lagrangian is invariant, is

$$\delta\phi = c, \quad \delta c = 0, \quad \delta d = b, \quad \delta b = 0. \tag{2.6}$$

The «physical» subspace is defined by $b^+|\Psi\rangle = 0$. On this subspace the Lagrangian as well as the Hamiltonian reduce to surface integrals over spatial infinity; this is no longer true after we introduce interactions and adopt our unconventional quantization.

In vector gauge theories the subspace of gauge modes is defined locally, as the space of exact vector fields. In singleton gauge theory the situation is very different, and this is a key point; all interpretation and all applications depend on it. The physical modes are distinguished from the gauge modes nonlocally, by boundary conditions at spatial infinity [4]. The field is therefore not a local observable, and this central feature is both a limitation and an opportunity. An important limitation is that any gauge invariant interaction is necessarily nonlocal (or «topological»); for example, of the type

$$\int j^\mu \partial_\mu \phi(x)\, dx$$

where the current is conserved. In a classical field theory such interactions are trivializable and thus of very limited interest, but in quantum field theory this is not necessarily the case, as will now be shown.

Consider the example of the interaction

$$\int \overline{\psi}\gamma^\mu(x)(\partial_\mu\phi)\psi(x)\, dx \tag{2.7}$$

with a Dirac field ψ . The Dirac equation

$$(\not{\partial} - im)\psi = -(\not{\partial}\phi)\psi$$

may be solved iteratively, $\psi = \psi_0 + \psi_1 + \psi_2 + \dots$,

$$(\partial\!\!\!/ - im)\psi_0 = 0,$$
$$(\partial\!\!\!/ - im)\psi_{n+1} = -(\partial\!\!\!/\psi)\psi_n, \quad n = 0, 1, \dots .$$

The solution up to ψ_2,

$$\psi_1(x) = -\phi(x)\psi_0(x)$$

$$\psi_2(x) = \frac{1}{2}\phi^2(x)\psi_0(x) - \int S_F(x, x')\gamma^\mu \frac{1}{2}[\phi(x'), \partial_\mu\phi(x')]\psi_0(x')\,\mathrm{d}\,x'.$$

In quantum field theory the interaction Lagrangian (2.7) needs to be regularized, one replaces it by the normal ordered product

$$\int : \overline{\psi}(x)(\partial\!\!\!/\phi)\psi(x) : \mathrm{d}\,x. \tag{2.8}$$

The definition of normal ordering depends on the choice of quantization, but is always carried out on free fields. The above expression makes sense in perturbation theory, the interpolating fields being expressed in terms of free fields. Our iteration scheme must now be carefully re-examined:

$$(\partial\!\!\!/ - im)\psi_0(x) = 0,$$

$$\begin{aligned}(\partial\!\!\!/ - im)\psi_1(x) &= -(\delta/\overline{\psi}(x))\int : \overline{\psi}(x')(\partial\!\!\!/\phi)\psi(x') : \mathrm{d}\,x'|_{\psi=\psi_0} = \\ &= -\int : \delta(x - x')(\partial\!\!\!/\phi)\psi_0(x') : \mathrm{d}\,x' = \\ &= - : \{\partial\!\!\!/\phi(x)\}\psi_0(x) := \\ &= : (\partial\!\!\!/ - im)\{\phi(x)\psi_0(x)\} : .\end{aligned}$$

Here we meet with a difficulty that we regard as a minor one, though we shall not endeavor to resolve it. We are dealing with a theory with interactions that involve derivatives. One knows that relativistic invariance is the result of cancellations between various quantities of doubtful signifiance. One knows also that the correct results can be obtained (in the framework of ordinary quantization) by naive application of Feynman rules. This amounts to the use of a modified time ordering ($T^\star$-product), that has the property of commuting with space and time differentiation. We feel justified, therefore, in expecting that the correct result is obtained here by moving the factor $\partial\!\!\!/ - im$ out of the embrace of the normal ordering. Thus $\psi_1 = -\phi\psi_0$. Next,

$$\begin{aligned}
(\partial\!\!\!/ - im)\psi_2(x) &= -(\delta/\delta\overline{\psi}(x)) \int :\overline{\psi}(x')(\partial\!\!\!/\phi)\psi(x'):\,\mathrm{d}x'|_{\psi=\psi_1} = \\
&= - : \{\partial\!\!\!/\phi(x)\}\psi_1(x) := \\
&= : \{\partial\!\!\!/\phi(x)\}\phi(x)\psi_0(x) := \\
&= \frac{1}{2}(:(\partial\!\!\!/ - im)\phi^2(x)\psi_0(x) : - \\
&- : [\phi(x), \partial\!\!\!/\phi(x)]\psi_0(x) :) = \\
&= (\partial\!\!\!/ - im) : \frac{1}{2}\phi^2(x)\psi_0(x) : + \\
&+ : \frac{1}{2}[\phi(x), \partial\!\!\!/\phi(x)]\psi_0(x) : .
\end{aligned}$$

In conventional field theory the second expression vanishes, and the complete iteration leads to

$$\psi(x) = e^{-\phi(x)}\psi_0(x).$$

That is, the interaction is trivialized (removed) by a local field transformation. Though the calculation is formal, it probably makes sense in perturbation theory.

Since we want the interaction to be nontrivial, we must look for alternative quantization schemes. The fact that the singleton field operator is not locally observable provides the opportunity to circumvent the limitations that are imposed on conventional field theories by microcausality.

The above formal calculations suggest that it may be possible to interpret

$$\frac{1}{2i} : [\phi(x), \partial_\mu\phi(x)] : \tag{2.9}$$

(suitably defined) as the electromagnetic potential quantum field operator. Unlike earlier attempts in this direction [10], our suggestion does not depend for its plausibility on unverifiable hypotheses involving non-perturbative or other mystical considerations. Quite on the contrary: we want the relationship between $\phi(x)$ and $A_\mu(x)$ to hold even for free fields. What makes such a connection natural, and this only in the context of singleton field theory, is the kinematical fact that all two-singleton states are massless.

The amazing fact that all states of two free singletons are massless was noticed long ago [1]. With ordinary Bose-Einstein quantization it seemed natural to interpret photons as states of two singletons, but it was difficult to understand how such «bound» states with vanishing binding energy could acquire ordinary Bose-Einstein statistical properties. This situation is at once altered if the expression (2.9) is interpreted as the electromagnetic potential; in fact, it will be shown that conventional statistics for photons is rigorously obtainable. We shall give up Bose-Einstein quantization for singletons and, with it, the restriction to symmetric states. Antisymmetric states, such as those created from the vacuum by the operator (2.9), are outside and orthogonal to this subspace, and these new states are interpreted as photons.

Our program, therefore, is to quantize singleton field theory according to new principles that allow for additional, antisymmetric two-particle states, in such a way as to make these additional excitations behave like ordinary photons. We shall recall how it was first shown [6] that this is actually feasible; then we shall discuss some shortcomings of that first scheme.

3. QUANTIZATION, PRELIMINARIES

Let us decompose the singleton field operator,

$$\phi(x) = \sum_j \phi^j(x)\,a_j + \text{h.c.}, \tag{3.1}$$

in terms of positive energy creation operators $a^{\star j}$ and annihilation operators a_j, without yet making any assumptions about their commutation relations. Let $|0\rangle$ designate a non degenerate vacuum state,

$$a_j|0\rangle = 0,$$

and consider the space spanned by

$$a^{\star j}a^{\star k}a^{\star l}\ldots|0\rangle. \tag{3.2}$$

The symmetric states

$$S(a^{\star j}a^{\star k}\ldots)|0\rangle$$

will be interpreted as multi-singleton states; the new states

$$[a^{\star j}, a^{\star k}]|0\rangle$$

as one-photon states. This is justified as follows. The states $a^{\star j}|0\rangle$ carry the singleton representation shown in Eq. (2.2); the states $a^{\star j}a^{\star k}|0\rangle$ carry the direct product $(2.2)\otimes$

(2.2) . We know also that the direct second power of the physical quotient $D(1/2,0)$ contains only massless representations [1]. Finally, we also know that the skew part of $(2.2)\otimes(2.2)$ contains the full Gupta-Bleuler triplet $D(3,0)\to D(2,1)\to D(3,0)$ of de Sitter electrodynamics [11]. This implies the existence of a basis of positive energy photon creation operators $b^{\star\alpha}$ and annihilation operators b_α, together with complex coefficients C^α_{jk}, such that

$$(3.3)\qquad [a_j, a_k] = iC^\alpha_{jk} b_\alpha, \quad [a^{\star j}, a^{\star k}] = i\overline{C}^\alpha_{jk} b^{\star\alpha}.$$

The group theoretical analysis assures us that the coefficients can be so chosen that the states $b^{\star\alpha}|0\rangle$ have the appropriate kinematical properties that allow them to be interpreted as photons, but the statistical properties of these photons depend on commutation relations that remain unspecified so far.

We had assumed, provisionally, in order to explore the simplest possibility first, that

$$(3.4)\qquad [a_j, b_\alpha] = 0, \quad [b_\alpha, b_\beta] = 0.$$

This would ensure that the states

$$b^{\star\alpha} b^{\star\beta} \ldots S(a^{\star j} a^{\star k} \ldots)|0\rangle$$

form a basis for the generalized Fock space spanned by the states (3.2). In this sense our generalized Fock space looks like an ordinary Fock space with two kinds of particles, but two crucial questions need to be answered. We need to know what happens when a_m is applied to (3.2); is our generalized Fock space invariant under the action of $\phi(x)$? One must also enquire into the structure of the commutator between b_α and $b^{\star\beta}$; can the usual relation

$$(3.5)\qquad [b_\alpha, b^{\star\beta}] = \delta^\beta_\alpha$$

be achieved?

To the demonstrate that the structure (3.3, 4, 5) actually exists, let $\widetilde{a}_j, \widetilde{a}^{\star k}$ be a set of conventional, canonical singleton annihilation and creation operators, and let $b_\alpha, b^{\star\beta}$ be a set of ordinary photon operators. In particular, this implies that

$$[\widetilde{a}_j, \widetilde{a}^{\star k}] = \delta^k_j, \quad [b_\alpha, b^{\star\beta}] = \delta^\beta_\alpha.$$

Suppose that the space of one-photon states carries the Gupta-Bleuler triplet representation

$$(3.6)\qquad D(3,0) \to D(2,1) \to D(3,0)$$

of so(3,2) ; then we know that there exist Clebsch-Gordan coefficients C^{α}_{jk} that intertwine this representation with the product of (2.2) with itself. Now let

$$a^{\star j} \equiv \widetilde{a}^{\star j} + \frac{i}{2}\overline{C^{\alpha}_{jk}}b^{\star\alpha}\widetilde{a}_k; \tag{3.7}$$

then Eqs. (3.3, 4) are satisfied. With this realization of the structure (3.3, 4, 5) the generalized Fock space spanned by the states (3.2) is invariant under the action of a_m and thus under the action of $\phi(x)$.

We are thus entitled to postulate the structure given by (3.3, 4, 5) and return to the proposed interpretation of (2.9) suitably defined, as the electromagnetic potential, in the case of free fields. Now if

$$A_\mu(x) \equiv \frac{1}{2i}\{:\phi(x)\partial_\mu\phi(x): - :\partial_\mu\phi(x)\phi(x):\}, \tag{3.8}$$

where the colons indicate a regularized product, so defined that the cross terms $a_j a^{\star k}$ in the two products cancel each other, then

$$A_\mu(x) = \sum_\alpha A^{\alpha}_\mu(x) b_\alpha + \text{h.c.},$$

$$A^{\alpha}_\mu(x) = \frac{1}{2}\sum_{j,k}\phi^j(x)\partial_\mu\phi^k(x)C^{\alpha}_{jk}.$$

This result does indeed justify the interpretation of (3.8). The effective interaction arising from $\int\overline{\psi}(\partial\!\!\!/\phi)\psi$ is $\int\overline{\psi}\,A\!\!\!/\psi$ and an argument based on Dyson's formula for the S-matrix in terms of free fields leads to the conclusion that the theory reduces effectively to QED [6]. However, this construction is incomplete in two respects.

First, the regularized product in (3.8) was not yet fully defined. In conventional field theory normal ordering does indeed lead to cancellation between the cross terms $a_j a^{\star k}$ in (3.8), leaving the other terms unaffected (and reducing the whole expression to zero). In the new framework we simply assumed that «normal ordering» gets rid of the cross terms while leaving all other terms in the written order. The proper definitions will be given below and constitute one of the principal results of this paper.

The second problem seems at first sight to be mainly an aesthetic one; it involves the incorporation of conformal invariance into the theory. The space of one-particle states of flat space QED involves two inequivalent irreducible representations of the Poincaré group, associated with the two helicities. The potential, however, is constructed in terms of a single indecomposable Gupta-Bleuler triplet. Conformal electrodynamics, in flat space as well as in de Sitter space, is described by the following indecomposable so(4,2) Gupta-Bleuler triplet [12]:

$$D(1,1/2,1/2) \to \{D(2,1,0)\oplus D(2,0,1)\oplus \mathrm{Id}\} \to D(1,1/2,1/2). \tag{3.9}$$

Reduction with respect to the de Sitter subalgebra reveals [13] two inequivalent so$(3,2)$ triplets, namely

$$(3.6) \qquad D(3,0) \to D(2,1) \to D(3,0)$$

$$(3.10) \qquad D(1,1) \to \{D(2,1) \oplus \mathrm{Id}\} \to D(1,1).$$

The fact that two copies of the physical representation $D(2,1)$ appear, corresponds to the two helicities in flat space but, unlike the situation in Minkowski space, they are not mixed up in the same triplet. This gives rise to an option in de Sitter space that is not available in flat space: one can construct a potential with the aid of just one (either one) of the two triplets, rather than using both of them. The price to pay for this would be loss of conformal invariance.

Now an apparent conflict between our construction of electrodynamics on the one hand, and conformal invariance on the other hand, seems to arise as follows. The kinematical basis for associating two singletons with one photon is that $D(2,1)$ appears as a subrepresentation of the direct product $D(1/2,0) \otimes D(1/2,0)$. But field theory has to take the gauge structure and the whole Gupta-Bleuler triplets into account. The unique singleton triplet is shown in Eq. (2.2); we used the fact that (3.6) appears as a summand in the reduction of the product of (2.2) with itself. The same is evidently not true of (3.10); this triplet does not appear in the direct product of two singleton triplets. Thus it would seem that our theory selects the option referred to above, available only in de Sitter space, and that conformal invariance is thus lost. We shall see that this is in fact not a valid conclusion.

4. SINGLETON QUANTIZATION ALGEBRA

Conformal invariance may seem like a secondary consideration, welcome if the cost is not too high, but perhaps not very fundamental. We shall see, however, that when we insist on the inclusion in QED of both triplets, (3.6) and (3.10), then we are rewarded by being led to a very satisfactory solution of the other problem discussed in Section 3, in connection with the definition of normal ordering.

The key point is that the second triplet cannot be accounted for by the states created from the vacuum by the operators $[a^{*j}, a^{*k}]$; but $[a_j, a^{*k}]$ *can* produce the second triplet. [Both statements need a good deal of explanation, for we are not dealing with unitary representations; please see Appendix]. One is thus led to associating new excitations with both of these operators

$$(4.1) \qquad [a_j, a_k] = iC^{\alpha}_{jk} b_{\alpha}, \quad [a_j, a^{*k}] = iC'^{k\alpha}_{j} b'_{\alpha},$$

such that (the linear span of) b^*_α carries (3.6) and b'_α carries (3.10) plus its contragredient. One hopes that the photon operators $b_\alpha, b^{*\alpha}$ and b'_α obey conventional Bose-

Einstein commutation relations and that they commute with the singleton operators a_j. These desiderata can essentially be satisfied.

Let us write a_{-j} for a^{*j}, $j = 1, 2, \ldots$ and let the indices $j, k, \ldots$ run over both positive and negative integers. Define operators b_{jk} as the commutator $[a_j, a_k]$. We observe that they may be viewed as constituting an operator valued generalization of the sympletic form that characterizes Bose-Einstein quantization; photons animate the originally rigid symplectic form just as gravitons give dynamical life to the metric tensor in general relativity. We want photons to be Bose-Einstein quanta, so we require that the commutator $[b_{jk}, b_{lm}]$ be a numerical tensor (numerical multiple of the unit operator). This tensor can be interpreted as a C-valued symplectic form, except that (as a two-form) it may be degenerate. The spectrum of photons is determined by the co-kernel, so we must insist that the choice of this form remain our privilege. To complete the algebraic structure we must also specify the commutator $[a_j, b_{kl}]$; evidently it cannot vanish as this would imply that the b's commute, by the Jacobi identity. We shall now demonstrate, by direct construction, that there exists a structure that incorporates all our desiderata and that is free of internal contradictions.

Let $(z_j, \overline{z}^j)$, $j = \pm 1, \pm 2, \ldots$ be a set of conventional canonical operators, with

$$[z_j, z_k] = 0 = [\overline{z}^j, \overline{z}^k],$$
$$[z_j, \overline{z}^k] = \delta_j^k$$

and let (B_{jk}), $j, k = \pm 1, \pm 2, \ldots$ be generators of a Heisenberg algebra, commuting with z_l and $\overline{z}^l$,

$$[B_{jk}, B_{lm}] = \varepsilon_{jk,lm}, \quad [B_{jk}, z_l] = 0 = [B_{jk}, \overline{z}^l] \tag{4.2}$$

in which ε is numerical. Though we call the B_{jk} the generators of a Heisenberg algebra, we do not mean to imply that the two-form $\varepsilon_{..,..}$ is non-degenerate. In this regard we reserve our options, about which more below. We use these operators to construct singleton operators,

$$a_j = z_j + B_{jk}\overline{z}^k. \tag{4.3}$$

This formula differs from Eq. (3.7) in the following respect. The operators $(\widetilde{a}_j)$, $j = 1, 2, \ldots$ in (3.7) are associated with a linear representation of $\mathrm{so}(3,2)$ that is equivalent to a positive energy singleton triplet (2.2), while the linear space spanned by (z_j), $j = \pm 1, \pm 2, \ldots$ carries the singleton triplet *plus* its negative energy contragredient. The $\overline{z}^j$ transform contragrediently and define a basis that is dual to that defined by the z_j. This implies that the B_{jk} transform like $a_j a_k$. The difficulties associated with Eq. (3.7) are here avoided because all the z_j commute with each other (whereas $\widetilde{a}_j$ and $\widetilde{a}_{-j} = \widetilde{a}^{*j}$

do not). We have doubled the space of building blocks, without increasing the number of relevant operators.

The operators B_{jk} are taken to commute with the z 's. Direct calculation now gives

$$[a_j, a_k] = B_{kj} - B_{jk} + \overline{z}^l \overline{z}_m \varepsilon_{jl,km} \equiv \omega_{jk} + b_{jk}. \tag{4.4}$$

The operator valued form defined by the commutator may be regarded as arising from an ordinary symplectic form by deformation. To make this explicit we have defined b_{jk} as the difference $[a_j, a_k] - \omega_{jk}$, where ω is the original, c-number symplectic form that characterizes the singleton triplet. To justify the inclusion of the numerical term ω_{jk}, we have to suppose that some of the B_{jk} are numerical. This is not in contradiction with (4.2), and it is one of the reasons why we needed to preserve our options with regard to the choice of ε.

With this definition we find

$$[b_{jk}, b_{lm}] = (\varepsilon_{jk,lm} - (j,k)) - (l,m) \equiv f_{jk,lm}. \tag{4.5}$$

The b_{jk} will be interpreted as photon operators and the symplectic form f will therefore be given a priori. For this reason it is necessary to verify that one can always find an ε that solves the last equation. In fact, a solution is

$$\varepsilon_{jk,lm} = (f_{jk,lm} - (k,m)) + (j,l).$$

With this choice of ε the term $\overline{z}\overline{z}\varepsilon$ in (4.4) vanishes.

We complete the algebraic structure by evaluating

$$[a_j, b_{kl}] = \{\varepsilon_{kj,lm} + \varepsilon_{jm,lk} - (k,l)\}\overline{z}^m. \tag{4.6}$$

The commutator algebra generated by the singleton operators a_j can therefore be made to close after the inclusion of b_{jk} and $\overline{z}^j$. The complete structure is

$$\begin{aligned} &[a_j, a_k] = \omega_{jk} + b_{jk}, && [a_j, \overline{z}_k] = \delta_j^k, \\ &[a_j, a_{kl}] = \Omega_{jklm}\overline{z}^m, && [b_{jk}, \overline{z}_l] = 0, \\ &[b_{jk}, b_{lm}] = f_{jk,lm}, && [\overline{z}_j, \overline{z}_k] = 0. \end{aligned} \tag{4.7}$$

This algebra characterizes singleton quantization just as the Heisenberg algebra is the basis for Bose-Einstein quantization.

5. FOCK SPACE

If we want to be quite precise, then we must point out that (4.7) is not an abstract Lie algebra, because of the appearance, in three places, of the unit operator. To get an abstract algebra we must introduce three new central elements and replace the 1st, 2nd and 5th equation by

$$(4.7') \qquad \begin{aligned} &[a_j, a_k] = \omega_{jk} e_1 + b_{jk}, \quad [a_j, \bar{z}^k] = \delta_j^k e_2, \\ &[b_{jk}, b_{lm}] = f_{jk,lm} e_3 . \end{aligned}$$

However, we shall limit ourselves to representations in which e_1, e_2 and e_3 are represented by the unit operator.

The next step is to choose a realization of the algebra (4.7). As always, the main physical input is the requirement of positivity of the energy. Recall that $a^{*j} = a_{-j}$ for $j > 0$ increases the energy (since the singleton triplet is a strictly positive energy representation) and that $a_j (j > 0)$ decreases it. The operators z_j carry the same representation as the a_j (the direct sum of the singleton triplet and its contragredient) and the z^{-j} carry the contragredient. We therefore require a unique vacuum such that $a_j|0\rangle = 0$ and $\bar{z}^{-j}|0\rangle = 0$ for $j = 1, 2, \ldots$. We also fix $e_1|0\rangle = e_2|0\rangle = e_3|0\rangle = |0\rangle$.

The operators b_{jk} for $j, k > 0$ have negative energy and therefore also annihilate the vacuum. The b_{jk} for j, k of opposite sign are photon creation and destruction operators associated with the second Gupta-Bleuler triplet (3.10) and its contragredient. The energy in (3.10) is not strictly positive, because of the presence of the one-dimensional subspace with zero energy. But this is quite irrelevant for our immediate purpose; the operators that create photons in (3.10) will therefore be called positive energy operators and the annihilation operators will be referred to as having negative energy.

With this understanding we complete our definition of the vacuum by requiring that all the negative energy operators (among the $a_j, b_{jk}, \bar{z}^j$) annihilate it, and that $e_a|0\rangle = |0\rangle$ for $a = 1, 2, 3$. [The z_j do not belong to the algebra, but are defined within the enveloping algebra by Eq. (4.3); it then follows that $z_j|0\rangle = 0$ for $j > 0$].

The energy lowering operators (annihilation operators), together with the central elements e_1, e_2, e_3, form a nilpotent subalgebra $\mathcal{B}$ of (4.7) [modified as in (4.7')]. Our Fock space representation is thus induced from the unique one-dimensional representation π_0 of this subalgebra, with character $\pi_0(e_a) = 1$, $a = 1, 2, 3$. This Fock space is generated by applying the operators a_j to the vacuum, the operators b_{jk} being expressed in terms of the a_j by (4.4). We cannot (unless we want to drop the second triplet) generate Fock space by using a_j 's with positive energy only, since $a_j a_k$, with a_k increasing the energy and a_j lowering it, would then have been left out. As for the $\bar{z}^j$, they appear only in the combination (4.6). In other words, Fock space is the direct product

$$U[a_j] \underset{\mathcal{B}}{\otimes} V_0,$$

where $U[a_j]$ is the free algebra generated by the a_j 's, modulo the commutation relations, $\mathcal{B}$ is the nilpotent algebra defined above, and V_0 is the one-dimensional vacuum space on which π_0 acts.

An involutive anti-automorphism of (4.7) is defined by $\star : a_j \to a^{\star j} = a^{-j}$. An (indefinite) inner product may be introduced on Fock space in the usual way, making $\star$ the formal adjoint.

6. FIELD COMMUTATORS

Let us now examine the free field commutators. We have

$$\phi(x) = \sum_j \phi^j(x) a_j$$

$$(6.1) \qquad \begin{aligned} [\phi(x), \phi(y)] &= \sum_{j,k} \phi^j(x)\phi^k(y)[a_j, a_k] = \\ &= \sum_{j,k} \phi^j(x)\phi^k(y)(\omega_{jk} + b_{jk}). \end{aligned}$$

Only the conventional, c-number term – involving ω – prevents this from being well defined in the limit $x \to y$. We therefore introduce a «normal ordering» precisely as in Bose-Einstein field theory.

Let Q be any polynomial in the $a_j, b_{jk}, \bar{z}^j$. There is a unique decomposition

$$(6.2) \qquad Q = \sum_{a,b} P_a^\star P_b C^{ab}[\omega, f],$$

in which the P_a are symmetric polynomials in the positive energy operators and the coefficients C^{ab} are polynomials in the components of the symplectic forms ω and f.

A polynomial of this type, but with coefficients that are independent of ω and f, is called a normal ordered polynomial. In particular,

$$(6.3) \qquad : Q := \sum_{a,b} P_a^\star P_b C^{ab}[0,0]$$

is said to be the normal ordered polynomial that corresponds to Q. We define a product on the space of normal ordered polynomials by

$$(: P :, : Q :) \to: (: P :)(: Q :) :=: PQ : .$$

This algebra of normal products is isomorphic to the enveloping algebra of the contracted Lie algebra

$$
\begin{aligned}
&[a_j, a_k] = b_{jk}, \quad [a_j, \overline{z}_k] = \delta_j^k, \\
(6.4) \qquad &[a_j, b_{kl}] = 0, \quad [b_{jk}, \overline{z}_l] = 0, \\
&[b_{jk}, b_{lm}] = 0, \quad [\overline{z}_j, \overline{z}_k] = 0
\end{aligned}
$$

which is a contraction of (4.6), but not Abelian.

In particular, we shall be concerned with the normal ordered commutator

$$: [\phi(x), \phi(y)] := \sum_{j,k} \phi^j(x)\phi^k(y) b_{jk},$$

and especially

$$: [\phi(x), \partial_\mu \phi(x)] := \sum_{j,k} \phi^j(x)\partial_\mu \phi^k(x) b_{jk}.$$

This last can evidently be identified with the free electromagnetic potential. If (b_α) is a basis for the photon operators, and

$$b_{jk} = C^\alpha_{jk} b_\alpha,$$

then

$$
\begin{aligned}
&\frac{1}{2i} : [\phi(x), \partial_\mu \phi(x)] := A_\mu(x) = \sum_\alpha A^\alpha_\mu(x) b_\alpha, \\
(6.5) \qquad &A^\alpha_\mu(x) = \frac{1}{2i} \sum_{j,k} \phi^j(x)\partial_\mu \phi^k(x) C^\alpha_{jk}.
\end{aligned}
$$

Besides this result, we shall need

$$
\begin{aligned}
&: [\phi(x), A_\mu(x)] := \frac{1}{2i} \sum_{jkl} \phi^j(x)\phi^k(x)\partial_\mu \phi^l(x) \ : [a_j, b_{kl}] := 0 \\
(6.6) \qquad &: [A_\mu(x), A_\nu(y)] := 0.
\end{aligned}
$$

7. COMPOSITE QED

We now return to the interaction (2.8) and the expression for ψ_2 :

$$
\begin{aligned}
\psi_2(x) = &\frac{1}{2}\phi^2(x)\psi_0(x) \\
&- \int S_F(x, x') : \frac{1}{2}[\phi(x'), \partial\!\!\!/ \phi(x')] : \psi_0(x') \, d x'.
\end{aligned}
$$

This strongly suggests identifying

$$:[\phi, \partial_\mu \phi]:/2i$$

with the electromagnetic field, up to second order of perturbation theory. To find the complete expression for $A_\mu(x)$ we shall look at the problem from a slightly different angle.

The starting point is the Lagrangian

$$\int \mathrm{d}x \; :\overline{\psi}(x)(\gamma^\mu D_\mu - im)\psi(x): . \tag{7.1}$$

The primitive connection is

$$D_\mu = \partial_\mu - \partial_\mu \phi.$$

Triviality of the interaction in the case of ordinary quantization can be «demonstrated» by carrying out a field transformation

$$\psi(x) = e^{-\phi(x)}\psi'(x).$$

The effect of this field transformation on the Lagrangian is to convert (7.1) to

$$\int \mathrm{d}x \; :\overline{\psi}'(x)(\gamma^\mu D'_\mu - im)\psi'(x):, \tag{7.2}$$

$$D'_\mu =: e^{\phi(x)}(\partial_\mu - \partial_\mu\phi)e^{-\phi(x)}: . \tag{7.3}$$

With Bose-Einstein quantization one obtains $D_\mu = \partial_\mu$. In general it is an infinite series

$$D_\mu = \partial_\mu + \frac{1}{2}:[\phi, \partial_\mu\phi]: + \frac{1}{3}:[\phi,[\phi,\partial_\mu\phi]]: + \ldots \equiv \partial_\mu + A_\mu(x).$$

We have an interaction Lagrangian that looks like that of electrodynamics. It involves electron fields and the vector field A_μ. Instead of the free Maxwell Lagrangian we have the singleton Lagrangian $\mathcal{L}_\phi$, with gauge fixing and ghosts. (Elsewhere [9] we have verified that the BRST structure of singleton gauge theory induces the BRST structure of the electromagnetic potential). To determine whether this theory includes QED, we have to investigate the properties of A_μ as a quantum field operator. More precisely, to calculate the S-matrix, we have to examine the asymptotic properties and deduce from them the free propagator. This, together with the form of interaction, which we know already, will fix the S-matrix.

The asymptotic properties of A_μ are determined by those of ϕ; we assume that ϕ tends to a free field. In that case the expression for A_μ simplifies, for all the normal ordered commutators vanish, with the sole exception of the first one. This remaining term is precisely the free electromagnetic potential, as we have seen in the preceding sections. In particular, the two-point function and the propagator are precisely those of de Sitter QED. It would be tempting, therefore, to conclude that our theory, with the Lagrangian (7.1), contains de Sitter electrodynamics, with some additional interactions at infinity.

Interactions at infinity are not always completely negligible; we have in mind the situation in QCD, where work of 't Hooft has shown the relevance of surface terms for CP violation and the $U(1)$ problem [14]. There, the surface terms become relevant because of the existence of an anomaly. We cannot be sure, on the basis of what has been done so far, that our theory is anomaly free. In view of the fact that CP violation is latent in it, as we shall explain later, it is of special interest to investigate the possibility that anomalies may also be present. We do not carry out such a study in this paper, but we shall at least make a preliminary investigation of the S-matrix.

It is not immediately obvious that Dyson's formula,

$$S = T^\star \exp i \int : \overline{\psi}(x) \partial\!\!\!/ \phi \psi(x) : \mathrm{d}^4 x, \tag{7.5}$$

for the perturbative S-matrix in terms of free field, is valid in the framework of a field theory with non-standard quantization rules. To prove that it is, we first determine the free Hamiltonian, or rather, the contribution H_0 that comes from the singleton field. We suppose that the basis $\{z_i\}$ is so chosen that each operator creates an eigenstate of the free energy from the vacuum; that is, each z_i is a creation operator or an annihilation operator for the free Hamiltonian. The entire Fock space can be generated by applying z 's and $\overline{z}$'s and B 's to the vacuum. Since these operators generate a Heisenberg algebra, it is obvious that H_0 is the usual sum of bilinears, with coefficients that are just the associated values of the $\mathrm{so}(3,2)$ energy operator. Nothing prevents us from expressing H_0 this way, since z_i can be considered as a convenient abbreviation for $a_i - B_{ij}\overline{z}^j$. It is now obvious that $[H_0, a_i] = \omega_i a_i$, where ω_i is the eigenvalue of the $\mathrm{so}(3,2)$ energy operator associated with a_i. Therefore, $[iH_0, \phi(x)]$ is the time derivative of the free field ϕ, with the time dependence determined by the classical field equation. Since H_0, expressed in terms of the z 's and the B 's, has the usual harmonic oscillator form, the free quantum field evolves according to the same equations of motion as the classical field. This is what is responsible for the fact that the interaction Hamiltonian, in the interaction representation, has the same form as the interaction Lagrangian, and this justifies Dyson's formula in the present case. We are therefore sure of (7.5), and rewrite it as

$$S = \sum \frac{i^n}{n!} T^\star \left[\int : \overline{\psi}(x) \; \partial\!\!\!/ \phi \psi(x) : \mathrm{d}\, x \right]^n .$$

Let us now recall the argument that was meant to show that this contains the S-matrix of de Sitter QED.

The properties of the electron fields are entirely conventional, so we can apply Wick's theorem to these factors, and consider separately each term. If, in any one of the factors

$$\int : \overline{\psi}(x)\ \not\partial\phi\psi(x)\ :\ \mathrm{d}\,x,$$

neither $\overline{\psi}$ nor ψ is contracted, then both are replaced by free field wave functions, and then an integration by parts shows that the matrix element vanishes (except for the possibility of an interaction at infinity). Consequently, every factor must appear with at least one contraction, as in

$$\int \overline{\overline{\psi}(x)\ \not\partial\phi\psi(x)\,\mathrm{d}\,x \int \overline{\psi}}(x')\ \not\partial\phi\psi(x')\,\mathrm{d}\,x' =$$
$$= \int \overline{\psi}(x)\{\not\partial\phi(x)\}S_F(x,x')\{\not\partial\phi(x')\}\psi(x')\,\mathrm{d}\,x\,\mathrm{d}\,x'.$$

Integration by parts gives

$$\int \overline{\psi}(x)\phi(x)\{\not\partial\phi(x)\}\psi(x)\,\mathrm{d}\,x.$$

A similar term arises from contracting $\overline{\psi}(x)$ with $\psi(x')$, and combining these two terms we get

$$\int \overline{\psi}(x)[\phi(x),\not\partial\phi(x)]\psi(x)\,\mathrm{d}\,x \stackrel{?}{=} \int \overline{\psi}(x)\gamma^\mu A_\mu(x)\psi(x).$$

From this we concluded that $2n$ factors of the interaction combine in pairs to give precisely the terms of order n in the S-matrix of QED. Let us now discuss the weak points in this argument.

1. Actually, the two factors ϕ and $\not\partial\phi$ appear inside the embrace of the ordering prescription $T^\star$. This is notoriously ambiguous, even in ordinary field theory, whenever points coincide. We note that the commutator appears even in conventional field theory, where it reduces to an infinite c-number. This meaningless quantity is always ignored, and this is one of the ways in which ordinary field theory is regularized. The $T^\star$ product is the sum of a contracted commutator (which is dropped) and a normal ordered commutator. The latter is zero in ordinary field theory but not in the one being discussed. The conclusion is that the reduction of the singleton S-matrix to that of QED needs to be completed by regularization, which might possibly uncover a non-trivial correction, of the nature of an anomaly.

2. In the original version of this theory it had been assumed that $\phi(x)$ commute with $A_\mu(x)$. Here it has been seen that this is not completely correct, since $:[a_j, b_{kl}]:$ is not zero but a linear combination of the gost operators $\overline{z}$. One therefore has to worry about a possible contribution of the type

$$[\phi(x), [\phi(x), \partial\!\!\!/\phi(x)]] \equiv \Phi(x).$$

(The normal ordered part of this vanishes). This free field is linear in the $\overline{z}$'s, which are spinless, therefore it is a gradient. A gradient correction to A_μ normally makes no contribution to the S-matrix, but one may fear an effect here because Φ is in most respects similar to the singleton field ϕ. But the commutators $[\Phi, \partial\!\!\!/\Phi]$ and $[\Phi, \partial\!\!\!/\phi]$ both vanish, so in fact this gradient does not make any contributions to the S-matrix. (A quite similar consideration shows that free singletons present in the initial state do not scatter from electrons or from photons).

To summarize: When interactions at infinity are set aside, and if no surprise appears during the process of regularization, then our singleton S-matrix reduces to that of de Sitter QED. Let us add that we hope that regularization will reveal an anomaly; here is why.

The basic interaction is a coupling of $\partial_\mu\phi$ to the electromagnetic current. For this to be CP invariant, we must take ϕ to be CP odd. But then A_μ becomes CP even, and invariance is lost. Of course, if the fundamental interaction had no physical manifestations other than electromagnetic, the no CP violation would be observed. But any physical implications coming from the self-interactions of singletons at infinity (such interactions were first discussed in Ref. 15) could result in observable CP violation. The term in the Lagrangian that contains these interactions is a surface term (a «topological» term), which reminds us of the θ-term discovered by 't Hooft [14] in QCD. It is difficult to understand how such interactions can be observed, but it appears that an anomaly can have not had much success in the past. It is to be hoped that, if it occurs here, then it will be in high enough order, so as to be seen only in very accurate measurements, such as may be possible only via observation of the $K, \overline{K}$ system).

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APPENDIX A

The physical photon representations of the Poincaré group, with the two helicities

± 1 , both have unique extensions to the conformal group [16]. These in turn remain irreducible when restricted to $so(3,2)$, the restriction being $D(2,1)$ in each case. Thus, in de Sitter space, instead of two inequivalent representations corresponding to the two helicities, photons carry two copies of one and the same unitary, irreducible representation.

In field theory things are more complicated, for the physical states are not associated with fields but instead with equivalence classes of fields. Thus the following nondecomposable representation of the conformal group is encountered

$$\{D(2,1,0) \oplus D(2,0,1)\} \rightarrow D(1,1/2,1/2),$$

where $D(1,1/2,1/2)$ is associated with the gauge modes. When this representation is restricted to $so(3,2)$ we find the two copies of $D(2,1)$, but now each one comes with a different set of gauge modes. We no longer have two copies of the same representation. Finally, quantization needs scalar modes as well, and the minimal representation of the conformal group that has the requisite symplectic structure is (3.9). Its reduction on $so(3,2)$ is

$$\begin{aligned}(3.9)|_{so(3,2)} &= (3.6) \oplus (3.10) \oplus \\ &\oplus D(1,0) \oplus D(1,0) \oplus D(2,0) \oplus D(2,0).\end{aligned}$$

The first two terms are the two $so(3,2)$ photon triplets. The last four must be included in a conformally invariant quantization scheme, but they do not contribute any additional physical states. It has been known for at least ten years [1] that the singleton representation $D(1/2,0)$ of $so(3,2)$ has the following remarkable property:

$$D(1/2,0) \otimes D(1/2,0) = \sum_{s=0,1,\dots} D(s+1,s). \tag{A.1}$$

The summands are all massless and include the physical photon representation $D(2,1)$. In field theory we must extend this to the Gupta-Bleuler triplets. Now we claim that

$$(2.2) \otimes (2.2) \supset (3.6). \tag{A.2}$$

[The representation on the right occurs as a direct summand in the reduction of the direct product]. One should be aware, however, that the reduction of tensor products of non-unitary representations is not so straightforward as in the unitary case, exemplified by (A.1). The framework that we use here is that of highest weight Harish Chandra modules or $(\mathfrak{g}, K)$ modules, finite linear combinations of states with fixed energy and angular momentum. The left side of (A.2) is algebraically equivalent, as a $(\mathfrak{g}, K)$ module, to a direct sum of highest weight $(\mathfrak{g}, K)$ modules, and the triplet on the right side is one of the summands.

The only irreducible, highest weight (positive energy) $(\mathfrak{g}, K)$ module that can extend $D(1/2,0)$ is $D(5/2,0)$, so the only possible singleton triplet is (2.2). The minimal energy is thus $1/2$, and the minimal energy in the direct square is 1. It is therefore clear that the other photon triplet, with minimal energy zero, cannot occur as summand in the reduction of the left side of (A.2). However, we claim that there is a sense in which it is included in the direct product of (2.2) with its negative energy contragradient. But this takes us outside the safe domain of highest weight modules, and since the factors are not unitary no standard decomposition exists.

If π_1 and π_2 are two finite dimensional representations of an algebra in spaces V_1 and V_2, and $\pi_2^\star$ is the contragradient of π_2, then the direct product $\pi_1 \otimes \pi_2^\star$ is canonically identified with the space $\mathcal{L}(V_1, V_2)$ of linear maps from V_1 to V_2. This is not quite true in the case of infinite dimensional representations. When we speak loosely, and say that

$$(2.2) \otimes (2.2)^\star \supset' (3.10) \tag{A.3}$$

what we really mean is that there is a space L of mappings of the singleton triplet into itself, such that the natural action of $so(3,2)$ on L is algebraically equivalent to the existence of the Clebsch-Gordan coefficients $C_j^{\prime k\alpha}$ in (4.1). The proof will be given in Appendix C. In Appendix B we explain, much more explicitly, the simple case when $\mathfrak{g} = sl(2,R)$.

APPENDIX B

In the usual Cartan-Weyl basis the highest weight representation $D(1/2)$ of $sl(2,R)$ is given by

$$\text{Basis } |m\rangle, \quad m = 1/2, 3/2, \ldots$$
$$H|m\rangle = m|m\rangle$$

$$E^+|m\rangle = (m + 1/2)|m+1\rangle$$
$$E^-|m\rangle = (m - 1/2)|m-1\rangle.$$

The contragredient $D(1/2)^\star$ is

$$H|-m\rangle = -m|-m\rangle$$
$$E^+|-m\rangle = -(m - 1/2)|-m+1\rangle$$
$$E^-|-m\rangle = -(m + 1/2)|-m-1\rangle.$$

In $D(1/2) \otimes D(1/2)^\star$ take the basis $|m,-n\rangle = |m > \otimes| -n\rangle$ and consider

$$\begin{aligned}\psi &= \sum_m C_m |m,-m\rangle \\ E^+\psi &= \sum_m (m+1/2)(C_m - C_{m-1})\,|m+1,-m\rangle \\ E^-\psi &= \sum_m (m+1/2)(C_{m+1} - C_m)\,|m,-m-1\rangle .\end{aligned}$$

There is (up to a constant factor) a unique series

$$\psi_0 = \sum |m,-m\rangle \tag{B.1}$$

that is annihilated by $H, E^\pm$.

Since $D(1/2)^\star$ is contragradient to $D(1/2)$, we can interpret $|-m\rangle$ as the linear functional $\langle m|$ given by

$$\langle m|n\rangle = \delta_{mn} .$$

Then $|m,-n\rangle$ is the linear map $|l\rangle \to \delta_{nl}|m\rangle$ and ψ_0 is the identity operator, in Dirac's notation

$$\psi_0 = \sum_m |m\rangle\langle m|$$

Being invariant, it gives the trivial representation Id of $sl(2,R)$. This does not mean that Id appears as a summand in the unitary reduction of $D(1/2) \otimes D(1/2)^\star$; this reduction is a direct integral and contains no discrete summands.

We now try to find maps that transform as the highest weight representation $D(1)$. The ground state must have energy 1,

$$\psi_1 = \sum_m C_1^m |m+1,-m\rangle, \tag{B.2}$$

and it must be annihilated by E^-,

$$E^-\psi_1 = \sum_m \{C_1^m(m+1/2) - C_1^{m-1}(m-1/2)\}|m,-m\rangle = 0 .$$

This gives $C_1^m = 0$, so we fail, there is no space of maps that transform according to $D(1)$. But if we require instead that

$$E^-\psi_1 \propto \psi_0 ,$$

then we are successful with

(B.3) $$C_1^m = 1.$$

This means that $D(1)$ is realized in terms of equivalence classes of maps, the zero-class being the multiples of the identity map. On the maps themselves we find the nondecomposable representation

$$D(1) \to \mathrm{Id}.$$

This can be expanded to the triplet

(B.4) $$\mathrm{Id} \to \{D(1) \otimes D(1)^*\} \to \mathrm{Id},$$

but the construction is left for the reader.

There are other ways to express these facts. If we realize $D(1/2)$ by holomorphic functions (induction),

$$|m\rangle = z^{m-1/2}, \quad H = \frac{1}{2} + z\frac{\partial}{\partial z},$$
$$E^+ = z + z^2\frac{\partial}{\partial z}, \quad E^- = \frac{\partial}{\partial z},$$

then the identity mapping becomes a function of two variables,

$$\psi_0 = \sum_m (z\zeta)^m = (1 - z\zeta)^{-1}$$

and the cyclic vector for (B.4) is

$$(1-x)^{-1}[\ln(1-x) + \text{const}.], \quad x = z\zeta.$$

The question that we must answer is whether the second photon triplet (3.10) can be realized as a space of maps of the singleton triplet (2.2) into itself.

APPENDIX C

We want to prove that there is a space of maps on the singleton module (2.2) such that the natural action of $so(3,2)$ on it is equivalent to the second triplet (3.10). Take three copies of the de Sitter cone, with coordinates u, v, y; $u^2 = v^2 = y^2 = 0$. Let V be a space of functions u carrying $D(5/2,0) \to D(1/2,0)$ (the gauge submodule vanishes on $u^2 = 0$), V^* a space of functions of v carrying the contragredient of this representation and W a space of one-forms depending on y that carries (3.10). We know that the three spaces exist and that each is characterized by a fixed degree of homogeneity, respectively $-1/2, -1/2$, and -1 [9].

Now consider the invariant « 3-point function»

$$v_\mu(v \cdot y)^{-1} (u \cdot v)^{-1/2}. \tag{C.1}$$

It can be expanded in a Fourier series that involves positive energy functions of u and y, and negative energy functions of v. [Actually, this expansion defines a generalized function and gives a precise meaning to the formal expression (C.1).] Explicit calculation reveals that the functions of u and v carry precisely $D(5/2,0) \to D(1/2,0)$ and its contragradient, respectively. Unfortunately, the one-forms depending on y carry not (3.10) but only the submodule $\mathrm{Id} \to D(1,1)$. This proves that this latter representation can be realized as a space of operators on the module $D(5/2,0) \to D(1/2,0)$, which is not what we had hoped to prove. The function (C.1) is the only invariant function of the requisite degrees (except we can interchange u, v), so this attempt fails.

Instead of (C.1) consider the function

$$v_\mu(y \cdot u)(v \cdot y)^{-2} (u \cdot v)^{-3/2}.$$

and expand as before. This time we find functions of u that carry $D(5/2,0) \to D(1/2,0)$, functions of v that carry $D(5/2,0)$ and one-forms depending on y that carry all of (3.10). This means that the second photon triplet can be realized in terms of maps from $D(5/2,0) \to D(1/2,0)$ to $D(5/2,0)$. In other words, speaking loosely, the second photon triplet is contained in $D(5/2,0) \otimes (2.2)^\star$.

Now it is not difficult to show, though we leave out the demonstration of it, that the extension of these methods from the de Sitter cone to the de Sitter hyperboloid gives the following result.

PROPOSITION. There exists a space of maps on the module (2.2), with range the gauge submodule, that carries the second photon triplet (3.10).

The meaning of this is that the creation operators for (3.10) can be obtained from $[a_i, a^{\star j}]$ $(i, j > 0)$, using a_j's from all of (2.2) but $a^{\star j}$'s from just the gauge submodule.

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